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ISOMETRIES OF L_p-SPACES ASSOCIATED WITH SEMIFINITE VON NEUMANN ALGEBRAS

BY

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ABSTRACT. The paper determines the structure of (classes of) linear isometries between L_p-spaces associated with semifinite normal faithful traces on von Neumann algebras, generalizing works of M. Broise and B. Russo. Also established are some auxiliary results on L_p norm inequalities which are of interest by themselves.

1. Introduction. Let ϕ (resp., ψ) be a semifinite normal faithful trace on a von Neumann algebra $\mathfrak{A}$ (resp., $\mathfrak{B}$), $L_p(\phi)$ (resp., $L_p(\psi)$) be the L_p-space associated with ϕ (resp., ψ) (cf. [1], [2], [3] or [4]), and Θ a linear isometry from $L_p(\phi)$ onto $L_p(\psi)$. In [5], Russo proved that if $\phi = \psi$ is finite on $\mathfrak{A} = \mathfrak{B}$ and if $p = 1$, then for each Θ there are a Jordan $*$-automorphism Φ of $\mathfrak{A}$, a positive selfadjoint $Z \in L_1(\phi)$ affiliated with the center of $\mathfrak{A}$, and a unitary $W \in \mathfrak{A}$ such that for any $X \in \mathfrak{A}$ we have $\Theta(X) = \Phi(X)ZW$ (strong product, cf. [2]). When $\mathfrak{A} = \mathfrak{B}$ is abelian and $\phi = \psi$, similar results hold for arbitrary $p \neq 2, p \in (1, \infty)$, and can be deduced from [7]. In [6] Broise showed that if $p = 2$ and if $\Theta[L_2^+(\phi)] = L_2^+(\psi)$ (equivalently $\Theta[L_2^+(\phi)] \subset L_2^+(\psi)$, where L_2^+ denotes the positive part of L_2), then there are uniquely a Jordan $*$-isomorphism Φ of $\mathfrak{A}$ onto $\mathfrak{B}$, and a positive selfadjoint operator $Z \in L_2(\psi)$ affiliated with the center of $\mathfrak{B}$ such that

$$\Theta(X) = \Phi(X)Z \quad \text{for every } X \in \mathfrak{A} \cap L_2(\phi).$$

It is the purpose of this paper to establish similar results for a rather general class of $(\mathfrak{A}, \mathfrak{B}, \phi, \psi, p, \Theta)$. More precisely, we shall prove in §3 the following two theorems.

THEOREM. *Suppose* $\Theta[\mathfrak{A}^+ \cap L_p(\phi)] = \mathfrak{B}^+ \cap L_p(\psi)$, $p \in [1, \infty)$. *Then there exist uniquely a Jordan $*$-isomorphism Φ of $\mathfrak{A}$ onto $\mathfrak{B}$, and a positive selfadjoint operator $Z \in L_p(\psi)$ affiliated with the center of $\mathfrak{B}$ such that*

$$\Theta(X) = \Phi(X)Z, \qquad X \in \mathfrak{A} \cap L_p(\phi).$$

THEOREM. *Suppose $\mathfrak{A}, \mathfrak{B}$ are finite (ϕ, ψ may be semifinite) and*

$$\Theta[\mathfrak{A} \cap L_p(\phi)] = \mathfrak{B} \cap L_p(\psi), \qquad p \in [1, \infty) \setminus \{2\}.$$

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Then there are a Jordan $*$*-isomorphism* Φ *of* $\mathfrak{A}$ *onto* $\mathfrak{B}$, *a positive selfadjoint operator* $Z \in L_p(\psi)$ *affiliated with the center of* $\mathfrak{B}$, *and a unitary* $W \in \mathfrak{B}$ *such that*

$$\Theta(X) = \Phi(X)ZW, \qquad X \in \mathfrak{A} \cap L_p(\phi).$$

Some other results in **[5]** and **[6]** are also generalized in §3 to the present context.

In order to obtain the above theorems, we shall prove in §2 some results on L_p norm inequalities which are of interest by themselves [cf. Theorems 1 and 2 of §2 below].

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2. L_p norm inequalities. In this section we shall establish some auxiliary results needed in §3, of which some special cases are known (cf. **[7]** and **[12]**).

THEOREM 1. *Let* $S,T \in \mathfrak{A}^+ \cap L_p(\phi)$.

(i) *Let* $p > 1$. *Then* $\phi(S^p + T^p) \leqslant \phi[(S + T)^p]$, *and equality holds iff* $ST = 0$.

(ii) *Let* $p > 1$. *Then* $\phi[(S + T)^p] \leqslant 2^{p-1}\phi(S^p + T^p)$, *and equality holds iff* $S = T$.

(iii) *Let* $p \in (0, 1)$. *Then* $\phi(S^p + T^p) \geqslant \phi[(S + T)^p]$. *If* $ST = 0$, *then we have equality. On the other hand, if we have equality, if* ϕ *is finite, and if* $S + T$ *either has a bounded inverse or is of finite rank, then* $ST = 0$.

(iv) *Let* $p \in (0, 1)$. *Then* $\phi[(S + T)^p] \geqslant 2^{p-1}\phi(S^p + T^p)$, *and equality holds iff* $S = T$.

REMARKS. (1) The inequalities obtained here for $S,T \in \mathfrak{A}^+ \cap L_p(\phi)$ can be easily extended to $S,T \in L_p^+(\phi)$ by using results in **[2]**. As to when equality holds [for $S,T \in L_p^+(\phi)$], the conditions are easily seen to be sufficient by **[4]**. Since we have not decided whether the conditions are also necessary and shall not use them in §3, we omit the details of such a generalization.

(2) It is plausible that for $p \in (0, 1)$ and $S,T \in \mathfrak{A}^+ \cap L_p(\phi)$, $\phi(S^p + T^p) = \phi[(S + T)^p]$ implies $ST = 0$. As evident from the proof of (ii) below, this assertion will follow from its special case where $S + T$ is injective.

We shall need the following lemmas in proving the above theorem. (**C**, **R**, **N** denote, respectively, the set of all complex, real, strictly positive integral numbers.)

LEMMA 1. *Let the map* $t \in I \mapsto S_t \in \mathfrak{A}^+ \cap L_p(\phi)$ *be differentiable* (*with respect to the norm on* $\mathfrak{A}$) *on the real open interval* I. *Suppose for some* $a \in I$, $S_a' \equiv dS_t/dt|_{t=a} \in \mathfrak{A} \cap L_p(\phi)$.

(i) *For any* $p \in [1, \infty)$ *we have*

$$\frac{d}{dt}\phi(S_t^p)\bigg|_{t=a} = \phi\big[pS_a^{p-1}S_a'\big]. \tag{1}$$

(ii) *Suppose* $p \in (0, 1)$. *If* ϕ *is finite and if* S_a *is invertible in* $\mathfrak{A}$, *then* (1) *holds.*

PROOF. Ad (i). Fix $p \in [1, \infty)$ and fix a compact subinterval I_1 of I with $a \in I_1$. Choose a branch of the complex function z^p, and a sector

$$\Omega \equiv \{re^{i\theta}: r \in [0, \infty), \theta \in (-\theta_1, \theta_1)\} \qquad (\theta_1 > 0)$$

of the right half-plane of $\mathbf{C}$ so that the branch is a single-valued holomorphic function on the sector. For $\varepsilon > 0$ let Γ_ε be the positively oriented contour consisting of the segments

$$\{re^{-i\theta_2}: r \in [\varepsilon, b)\}, \qquad \{re^{i\theta_2}: r \in [\varepsilon, b)\}$$

and the circular arc $\{be^{i\theta}: \theta \in [-\theta_2, \theta_2]\}$, where $\theta_2 \equiv \theta_1/2$, and $b > \operatorname{Max}\{\|S_t\|: t \in I_1\}$. Let Γ be similarly defined with ε replaced by zero. Since (cf. [**8**, p. 272]) for any $\lambda \in \Gamma \setminus \{0\}$,

$$\|\lambda^p(\lambda - S_t)^{-1}\| < |\lambda|^p\,|\operatorname{Im}\lambda|^{-1} = |\lambda|^{p-1}\sin^{-1}\theta_2$$

($\operatorname{Im}\lambda \equiv$ imaginary part of λ), we see that $\int_{\Gamma_\varepsilon}\lambda^p(\lambda - S_t)^{-1}\,d\lambda$ converges in norm as $\varepsilon \to 0$; we shall denote the limit by $\int_\Gamma \lambda^p(\lambda - S_t)^{-1}\,d\lambda$. It is then easy to see that, for every $t \in I_1$,

$$S_t^p = \frac{1}{2\Pi i}\int_\Gamma \lambda^p(\lambda - S_t)^{-1}\,d\lambda.$$

Now the above construction works for the function z^{p-1} as well, provided that the sector Ω and Γ have been chosen carefully. Thus we also have

$$S_t^{p-1} = \frac{1}{2\Pi i}\int_\Gamma \lambda^{p-1}(\lambda - S_t)^{-1}\,d\lambda.$$

Similarly we see that

$$(2\Pi i)^{-1}\int_{\Gamma_\varepsilon}\lambda^p(\lambda - S_a)^{-1}S_a'(\lambda - S_a)^{-1}\,d\lambda$$

converges in norm as $\varepsilon \to 0$; we denote the limit by D. Now we claim that the map $t \to S_t^p$ is differentiable (with respect to the norm) and

$$\frac{d}{dt}S_t^p\bigg|_{t=a} = D. \tag{2}$$

In fact we have

$$\|(t-a)^{-1}(S_t^p - S_a^p) - D\| < l_1(t) + l_2(t)$$

where

$$l_1(t) \equiv \lim_{\varepsilon\to 0}(2\Pi i)^{-1}\left\|\int_{\Gamma_\varepsilon}\lambda^p(\lambda - S_a)^{-1}(D_t - S_a')(\lambda - S_a)^{-1}\,d\lambda\right\|,$$

$$l_2(t) \equiv \lim_{\varepsilon\to 0}(2\Pi i)^{-1}\left\|\int_{\Gamma_\varepsilon}\lambda^p\left[(\lambda - S_t)^{-1} - (\lambda - S_a)^{-1}\right]D_t(\lambda - S_a)^{-1}\,d\lambda\right\|$$

with $D_t \equiv (t-a)^{-1}(S_t - S_a)$. It is easy to see that $\lim_{t\to a} l_1(t) = 0$. For $l_2(t)$, choose a constant k and a compact subinterval I_2 of I with $a \in I_2$ and

$$\sup\{\|D_t\| : t \in I_2, t \neq a\} < k.$$

For $\varepsilon_1 > 0$ and $t \in I_2 \setminus \{a\}$ we have

$$\begin{aligned}&\left\|\int_{\Gamma_{\varepsilon_1}}\lambda^p\left[(\lambda - S_t)^{-1} - (\lambda - S_a)^{-1}\right]D_t(\lambda - S_a)^{-1}\,d\lambda\right\|\\ &\qquad = \left\|\int_{\Gamma_{\varepsilon_1}}\lambda^p(\lambda - S_t)^{-1}(S_t - S_a)(\lambda - S_a)^{-1}D_t(\lambda - S_a)^{-1}\,d\lambda\right\|\\ &\qquad < m_{\varepsilon_1}k\|S_t - S_a\|\end{aligned}$$

where m_{ε_1} is a constant which depends on ε_1 only. On the other hand, for $0 < \varepsilon < \varepsilon_1$,

$$\begin{aligned}&\left\|\int_{\Gamma_\varepsilon\setminus\Gamma_{\varepsilon_1}}\lambda^p\left[(\lambda - S_t)^{-1} - (\lambda - S_a)^{-1}\right]D_t(\lambda - S_a)^{-1}\,d\lambda\right\|\\ &\qquad < 4k\int_\varepsilon^{\varepsilon_1} r^{p-2}\left[\sin\theta_2\right]^{-2}dr < 4k\varepsilon_1^{p-1}\left[(p-1)\sin^2\theta_2\right]^{-1}.\end{aligned}$$

From these estimates it is clear that $\lim_{t\to a} l_2(t) = 0$. Thus our claim (2) is established.

By the spectral theorem and the semifiniteness of ϕ, there exists an increasing sequence $(E_n)_{n\in\mathbf{N}}$ of projections in $\mathfrak{A}$ with $\phi(E_n) < \infty$, $E_nS_a = S_aE_n$, and $\lim_{n\to\infty} E_n = I$. But we have in the sense of norm convergence:

$$\left(\frac{d}{dt}S_t^p\Big|_{t=a}\right)E_n = (2\Pi i)^{-1}\int_\Gamma \lambda^p(\lambda - S_a)^{-1}S_a'(\lambda - S_a)^{-1}E_n\,d\lambda.$$

Hence

$$\phi\left[\left(\frac{d}{dt}S_t^p\Big|_{t=a}\right)E_n\right] = (2\Pi i)^{-1}\int_\Gamma \lambda^p\phi\left[(\lambda - S_a)^{-2}S_a'E_n\right]d\lambda.$$

Now, since

$$\frac{d}{d\lambda}\left[\lambda^p(\lambda - S_t)^{-1}\right] = p\lambda^{p-1}(\lambda - S_t)^{-1} - \lambda^p(\lambda - S_t)^{-2}, \qquad \lambda \in \Gamma_\varepsilon,$$

we have

$$pS_t^{p-1} = (2\Pi i)^{-1}\int_\Gamma p\lambda^{p-1}(\lambda - S_t)^{-1}\,d\lambda$$
$$= (2\Pi i)^{-1}\int_\Gamma \lambda^p(\lambda - S_t)^{-2}\,d\lambda,$$

and

$$\phi\left(pS_a^{p-1}S_a'E_n\right) = (2\Pi i)^{-1}\int_\Gamma \lambda^p\phi\left[(\lambda - S_a)^{-2}S_a'E_n\right]d\lambda,$$

i.e.,

$$\phi\left[\left(\frac{d}{dt}S_t^p\Big|_{t=a}\right)E_n\right] = \phi\left(pS_a^{p-1}S_a'E_n\right).$$

Letting $n \to \infty$ we obtain by (2)

$$\frac{d}{dt}\phi(S_t^p)\Big|_{t=a} = \phi\left(\frac{d}{dt}S_t^p\Big|_{t=a}\right) = \phi\left(pS_a^{p-1}S_a'\right).$$

Ad (ii). For t sufficiently close to a, S_t is invertible in $\mathfrak{A}$ so that the contour Γ can be replaced by one lying in a domain of analyticity for a branch of the complex function z^p, and containing the spectrum of such S_t in its interior. Thus our assertion follows from a well-known result [9, p. 108].

LEMMA 2. (i) *Let* $p \in (0, \infty)$ *and let* $A_1, A_2 \in \mathfrak{A}^+ \cap L_p(\phi)$ *with* $A_1 \geqslant A_2$. *Then* $\phi(A_1^p) \geqslant \phi(A_2^p)$.

(ii) *Let* $p \in (1, \infty)$, $n \in \mathbf{N}$ *and let* $A,B \in \mathfrak{A}^+ \cap L_{np}(\phi)$. *Denote* $C_p \equiv B^{1/2}(A + B)^{p-1}B^{1/2}$. *Then* $\phi(C_p^n) \geqslant \phi(B^{np})$.

(iii) *Suppose* ϕ *is finite and let* $A,B \in \mathfrak{A}^+$ *be such that* $A + B$ *is invertible in* $\mathfrak{A}$. *Then, for any* $p \in (0, 1)$,

$$\phi\left[(A + B)^{p-1}B\right] \leqslant \phi(B^p).$$

PROOF. Ad (i). We prove first that the assertion is valid for $p = 2^n$, $n \in \mathbf{N}$. Indeed this can be shown by induction on n with the following observations:

$$A_1^{2^n} = \left(A_1^{1/2}A_1A_1^{1/2}\right)^{2^{n-1}}, \qquad \phi\left[\left(A_1^{1/2}A_2A_1^{1/2}\right)^{2^n}\right] = \phi\left[\left(A_2^{1/2}A_1A_2^{1/2}\right)^{2^n}\right].$$

Now suppose $p \in (0, \infty)$ and choose $n \in \mathbf{N}$ so large that $p_1 \equiv p2^{-n} < 1$. Then

$$\phi(A_1^p) = \phi\left[(A_1^{p_1})^{2^n}\right] \geqslant \phi\left[(A_2^{p_1})^{2^n}\right] = \phi(A_2^p).$$

Ad (ii). We proceed by induction. If $p \in (1, 2]$, then $C_p \geqslant B^p$; so by (i) we have $\phi(C_p^n) \geqslant \phi(B^{np})$ for all $n \in \mathbf{N}$ and for all $A,B \in \mathfrak{A}^+ \cap L_{np}(\phi)$. Now suppose that for some integral $m \geqslant 2$, (ii) is valid for all $p \in (1, m]$ and all $n \in \mathbf{N}$. Let $l = m + \alpha$ with $\alpha \in (0, 1]$, $n \in \mathbf{N}$ and let $A,B \in \mathfrak{A}^+ \cap L_{nl}(\phi)$. Then $C_l \equiv B^{1/2}(A + B)^{l-1}B^{1/2} \geqslant D^2$ where $D \equiv B^{1/2}(A + B)^{l/2-1}B^{1/2}$.

Thus by (i), $\phi(C_l^n) \geqslant \phi(D^{2n})$. But $l/2 \in (1, m]$; by the induction hypothesis $\phi(D^{2n}) \geqslant \phi(B^{nl})$. Hence $\phi(C_l^n) \geqslant \phi(B^{nl})$ and (ii) is established.

Ad (iii). For a strictly positive number ε, let $B_\varepsilon \equiv \varepsilon + B$. Then

$$(A + B_\varepsilon)^{p-1} \leqslant B_\varepsilon^{p-1}, \qquad B_\varepsilon^{1/2}(A + B_\varepsilon)^{p-1}B_\varepsilon^{1/2} \leqslant B_\varepsilon^p.$$

Letting $\varepsilon \to 0$ we obtain $B^{1/2}(A + B)^{p-1}B^{1/2} \leqslant B^p$; so

$$\phi\left[(A + B)^{p-1}B\right] \leqslant \phi(B^p).$$

PROOF OF THEOREM 1. Ad (i). Define the function g on $\mathbf{R}^+$ by

$$g(t) \equiv \phi\left[(S + tT)^p\right] - \phi(S^p + t^pT^p), \qquad t \in \mathbf{R}^+.$$

Then $g(0) = 0$, and by the preceding lemmas (also [**1**, Corollary 3 of Theorem 6]),

$$g'(t) = \phi\left[pT^{1/2}(S + tT)^{p-1}T^{1/2}\right] - \phi(pt^{p-1}T^p) \geqslant 0, \qquad t \in \mathbf{R}^+.$$

Thus $g(1) \geqslant g(0)$, i.e., $\phi[(S + T)^p] \geqslant \phi(S^p + T^p)$. We proceed to demonstrate the assertion on the equality. If $ST = 0$, then obviously $TS = ST = 0$. Appealing to well-known commutative results we have $\phi[(S + T)^p] = \phi(S^p + T^p)$.

Suppose now $\phi[(S + T)^p] = \phi(S^p + T^p)$. To show $ST = 0$ we let

$$K \equiv i\left[T(S + T)^{p-1} - (S + T)^{p-1}T\right]$$

(which is a selfadjoint element of $\mathfrak{A}$), and define the function h on $\mathbf{R}$ by:

$$h(t) \equiv \phi\left[(S + e^{-itK}Te^{itK})^p\right], \qquad t \in \mathbf{R}.$$

Then the result we just obtained implies that h attains its absolute minimum at $t = 0$. By Lemma 1, we have

$$\begin{aligned} 0 = ip^{-1}h'(0) &= \phi\left[(S + T)^{p-1}(KT - TK)\right] \\ &= \phi\left\{\left[T(S + T)^{p-1} - (S + T)^{p-1}T\right]K\right\}. \end{aligned}$$

Hence $T(S + T)^{p-1} = (S + T)^{p-1}T$. By the spectral theorem, $T(S + T) = (S + T)T$, i.e., $ST = TS$. But for any two nonnegative real numbers s,t: $(s + t)^p = s^p + t^p$ only if $st = 0$. As we have $\phi[(S + T)^p] = \phi(S^p + T^p)$ where S and T commute, we conclude that $ST = 0$.

Ad (ii). By Minkowski's inequality,

$$\phi\left[(S + T)^p\right] \leqslant \left[\phi(S^p)^{1/p} + \phi(T^p)^{1/p}\right]^p.$$

But as $p > 1$, we have, for any $s,t \in \mathbf{R}^+$, $(s + t)^p \leqslant 2^{p-1}(s^p + t^p)$. Hence

$$\phi\left[(S + T)^p\right] \leqslant 2^{p-1}\phi(S^p + T^p).$$

If $S = T$ then obviously $\phi[(S + T)^p] = 2^{p-1}\phi(S^p + T^p)$. Suppose that $\phi[(S + T)^p] = 2^{p-1}\phi(S^p + T^p)$ for some $S,T \in \mathfrak{A}^+ \cap L_p(\phi)$. In order to show that $S = T$, we define a function f on $\mathbf{R}$ by

$$f(t) \equiv \phi\left[(S + e^{-itH}Te^{itH})^p\right], \qquad t \in \mathbf{R},$$

where H is a selfadjoint element of $\mathfrak{A}$. By the result we just obtained, f attains its absolute maximum at $t = 0$. Therefore

$$0 = ip^{-1}f'(0) = \phi\left[(S + T)^{p-1}(HT - TH)\right].$$

As before, we deduce that $ST = TS$. Since we have

$$\phi\left\{\left[2^{-1}(S + T)\right]^p\right\} = 2^{-1}\left[\phi(S^p) + (T^p)\right],$$

we conclude that $S = T$.

Ad (iii). Suppose at first that $S + T > \varepsilon$ (a positive number) and that ϕ is finite. Define a function g_1 on $\mathbf{R}^+$ by

$$g_1(t) \equiv \phi\left[(S + tT)^p\right] - \phi(S^p + t^pT^p), \qquad t \in \mathbf{R}^+.$$

Then $g_1(0) = 0$, and by the preceding lemmas, we have $g_1'(t) \leqslant 0$ for all $t \in (0, 1)$. Hence $g_1(1) \leqslant g_1(0)$, i.e.,

$$\phi\left[(S + T)^p\right] \leqslant \phi(S^p) + \phi(T^p).$$

The extra condition "$S + T > \varepsilon$ and φ is finite" can now be removed as follows. For given $S, T \in \mathfrak{A}^+ \cap L_p(\phi)$, let $(E_n)_{n\in\mathbf{N}}$ be an increasing sequence of projections in $\mathfrak{A}$ such that each E_n commutes with $S + T$, $(S + T)E_n \geqslant n^{-1}E_n$, and $(S + T)(I - E_n) < n^{-1}(I - E_n)$. As $E_n \leqslant n^p(S + T)^p$, we have $\phi(E_n) < \infty$. Then the result of the preceding paragraph implies that

$$\phi\left\{\left[(S + T)E_n\right]^p\right\} \leqslant \phi\left[(E_nSE_n)^p + (E_nTE_n)^p\right],$$

i.e.,

$$\left\|(S + T)^{1/2}E_n\right\|_{2p}^{2p} \leqslant \left\|S^{1/2}E_n\right\|_{2p}^{2p} + \left\|T^{1/2}E_n\right\|_{2p}^{2p}.$$

Taking adjoints we get

$$\left\|E_n(S + T)^{1/2}\right\|_{2p}^{2p} \leqslant \left\|E_nS^{1/2}\right\|_{2p}^{2p} + \left\|E_nT^{1/2}\right\|_{2p}^{2p},$$

i.e.,

$$\phi\left\{\left[(S + T)^{1/2}E_n(S + T)^{1/2}\right]^p\right\} \leqslant \phi\left[\left(S^{1/2}E_nS^{1/2}\right)^p\right] + \phi\left[\left(T^{1/2}E_nT^{1/2}\right)^p\right].$$

Since

$$\left(S^{1/2}E_nS^{1/2}\right)^p \leqslant S^p, \qquad \left(T^{1/2}E_nT^{1/2}\right)^p \leqslant T^p,$$

since E_n commutes with $S + T$, and since $(S + T)(I - \lim_{n\to\infty} E_n) = 0$, we have

$$\phi\left[(S + T)^p\right] = \lim_{n\to\infty} \phi\left\{\left[(S + T)^{1/2}E_n(S + T)^{1/2}\right]^p\right\} \leqslant \phi(S^p) + \phi(T^p).$$

We proceed to prove the assertion on the equality. If $ST = 0$, then obviously $\phi[(S + T)^p] = \phi(S^p + T^p)$. Suppose now that $\phi(S^p + T^p) = \phi[(S + T)^p]$

and that ϕ is finite. Assume at first that $S + T$ has a bounded inverse. Then by arguments similar to those used in proving (i), we have $ST = 0$. Assume now that $S + T$ is of finite rank. Let E_1 be the support of $S + T$, $E_2 \equiv I - E_1$, $S_1 \equiv E_1SE_1$ and $T_1 \equiv E_1TE_1$. Then $E_2SE_2 = E_2TE_2 = 0$, $S^{1/2}E_2 = T^{1/2}E_2 = 0$, $S_1 = (S^{1/2}E_1)^*(S^{1/2}E_1) = S$ and $T_1 = T$. Thus

$$\phi\left[(S_1 + T_1)^p\right] = \phi(S_1^p + T_1^p).$$

As $S_1 + T_1 = (S + T)E_1$ is injective and of finite rank, $S_1 + T_1$ has a bounded inverse. Thus $ST = S_1T_1 = 0$.

Ad (iv). As $p \in (0, 1)$, the map $A \mapsto A^p$ on $\mathfrak{A}^+$ is concave and we have

$$S^p + T^p \leqslant 2^{1-p}(S + T)^p, \qquad \phi\left[(S + T)^p\right] \geqslant 2^{p-1}\phi(S^p + T^p).$$

If $S = T$ then we obviously have equality. On the other hand, if we have equality $\phi[(S + T)^p] = 2^{p-1}\phi(S^p + T^p)$, then $(S + T)^p = 2^{p-1}(S^p + T^p)$. So $S + T = 2^{1-q}(S^p + T^p)^q$ where $q \equiv p^{-1}$. Let $S_1 \equiv S^p$, $T_1 \equiv T^p$. Then

$$\begin{aligned}\phi(S + T) &= 2^{1-q}\phi\left[(S_1 + T_1)^q\right]\\ &\leqslant 2^{1-q}2^{q-1}\phi(S_1^q + T_1^q) = \phi(S + T)\end{aligned}$$

by (ii) as $q > 1$. Hence

$$\phi\left[(S_1 + T_1)^q\right] = 2^{q-1}\phi(S_1^q + T_1^q).$$

By (ii) again, $S_1 = T_1$; so $S = T$. This completes the proof of Theorem 1.

THEOREM 2. *Let* $A,B \in L_p(\phi) \cap \mathfrak{A}$ ($\|\cdot\|_p$ *the* L_p *norm*).

(i) *If* $p \in [2, \infty)$, *then*

$$\|A + B\|_p^p + \|A - B\|_p^p \geqslant 2(\|A\|_p^p + \|B\|_p^p).$$

(ii) *If* $p \in [1, 2]$, *then*

$$\|A + B\|_p^p + \|A - B\|_p^p \leqslant 2(\|A\|_p^p + \|B\|_p^p).$$

(iii) *Suppose* $p \in [1, \infty)\setminus\{2\}$. *Then*

$$\|A + B\|_p^p + \|A - B\|_p^p = 2(\|A\|_p^p + \|B\|_p^p)$$

iff $AB^* = B^*A = 0$.

REMARK. Statement (i) is Lemma 3.5 of [**1**] in slightly different form. Statement (ii) can be obtained via Lemma 3.4 of [**1**] as Dixmier did for Lemma 3.5 of [**1**]. In fact they hold for any $A,B \in L_p(\phi)$. It suffices, therefore, to give:

PROOF OF THEOREM 2(iii). Suppose that $A,B \in \mathfrak{A} \cap L_p(\phi)$ satisfy

$$\|A + B\|_p^p + \|A - B\|_p^p = 2(\|A\|_p^p + \|B\|_p^p).$$

Let $q \equiv p/2$, $C \equiv A^*A + B^*B$, and $D \equiv A^*B + B^*A$. Then we have

$$\phi\left[(C + D)^q + (C - D)^q\right] = 2\phi\left[(A^*A)^q + (B^*B)^q\right].$$

By Theorem 1 we have for $q > 1$,

$$\phi\left[(C + D)^q + (C - D)^q\right] \geqslant 2\phi(C^q) \geqslant 2\phi\left[(A^*A)^q + (B^*B)^q\right],$$

and for $q \in (0, 1)$,

$$\phi\left[(C + D)^q + (C - D)^q\right] \leqslant 2\phi(C^q) \leqslant 2\phi\left[(A^*A)^q + (B^*B)^q\right].$$

Thus we have $\phi[(C + D)^q + (C - D)^q] = 2\phi(C^q)$ where $q \in (0, \infty)\setminus\{1\}$. By Theorem 1 (ii), (iv), $0 = D \equiv A^*B + B^*A$, i.e., $A^*B = -B^*A$. Since the equality

$$\|A + B\|_p^p + \|A - B\|_p^p = 2(\|A\|_p^p + \|B\|_p^p)$$

remains unchanged if (A, B) is replaced by (B^*, A^*), we have $BA^* = -AB^*$ also. Then we have

$$A^*AB^*B = -A^*BA^*B = A^*BB^*A = B^*BA^*A,$$

i.e., A^*A commutes with B^*B. Since

$$\phi\left[(A^*A + B^*B)^q\right] = \phi(C^q) = \phi\left[(A^*A)^q + (B^*B)^q\right]$$

and since, for nonnegative real numbers s, t, $(s + t)^q = s^q + t^q$ only if $st = 0$, we conclude that $A^*AB^*B = 0$. Hence $(BA^*A)^*(BA^*A) = 0$, $BA^* = 0$ on the support of A^* and $AB^* = 0$. Replacing (A, B) by (B^*, A^*) we have $B^*A = 0$ also

Suppose $A, B \in \mathfrak{A} \cap L_p(\phi)$ satisfy $AB^* = B^*A = 0$. Then A^*A commutes with B^*B, $(A^*A)(B^*B) = 0$ and $AB^* + B^*A = 0$. Hence

$$\begin{aligned}2\phi\left[(A^*A)^q + (B^*B)^q\right] &= 2\phi\left[(A^*A + B^*B)^q\right]\\ &= \phi\left\{\left[(A + B)^*(A + B)\right]^q + \left[(A - B)^*(A - B)\right]^q\right\},\end{aligned}$$

i.e.,

$$2(\|A\|_p^p + \|B\|_p^p) = \|A + B\|_p^p + \|A - B\|_p^p.$$

This completes the proof.

3. L_p isometries. In this section we shall prove our main theorems introduced in §1, and also generalizations of some results of **[5]** and **[6]**, by modifying the arguments of these papers. For clarity, however, we shall present the auxiliary results in full, though at places we shall refer the reader to **[6]** for more details.

Throughout this section, as before, we let ϕ [resp., ψ] be a semifinite normal faithful trace on $\mathfrak{A}$ [resp., $\mathfrak{B}$], $L_p(\phi)$ [resp., $L_p(\psi)$] be the associated L_p space, and Θ a linear isometry from $L_p(\phi)$ into $L_p(\psi)$, $p \in [1, \infty)\setminus\{2\}$. (As in §1, the sum and product of two measurable unbounded operators will be in the strong sense; cf. **[2]**.)

LEMMA 3. (i) *Let $(A_\lambda)_{\lambda\in\Lambda}$ be a net in $\mathfrak{A}$, which converges to $A \in \mathfrak{A}$ ultraweakly. Then for any $X \in L_p(\phi)$, $(A_\lambda X)_{\lambda\in\Lambda}$ [resp., $(XA_\lambda)_{\lambda\in\Lambda}$] converges to AX (resp., XA) with respect to the $\sigma(L_p(\phi),L_q(\phi))$ topology on $L_p(\phi)$, where $q^{-1}+p^{-1}=1$.*

(ii) *Let $(A_\lambda)_{\lambda\in\Lambda}$ be a net in $\mathfrak{A}^+ \cap L_p(\phi)$ bounded above by $A \in \mathfrak{A}^+ \cap L_p(\phi)$. Suppose $(A_\lambda)_{\lambda\in\Lambda}$ converges to A weakly and suppose $\phi(S) < \infty$ where S denotes the support of A. Then for any $B \in \mathfrak{B}$, $(\Theta(A_\lambda)B)_{\lambda\in\Lambda}$ [resp., $(B\Theta(A_\lambda))_{\lambda\in\Lambda}$] converges to $\Theta(A)B$ [resp., $B\Theta(A)$] with respect to the $\sigma(L_p(\psi),L_q(\psi))$ topology on $L_p(\psi)$.*

PROOF. Ad (i). Let $X' \in L_q(\phi)$. Then $XX' \in L_1(\phi)$ and $\phi(A_\lambda XX')$ converges to $\phi(AXX')$ by [9, p. 82]. Hence $(A_\lambda X)_{\lambda\in\Lambda}$ converges to AX with respect to the $\sigma(L_p(\phi),L_q(\phi))$ topology on $L_p(\phi)$. Similarly by [4, Lemma 3.1] we prove that (XA_λ) converges to XA with respect to the $\sigma(L_p(\phi),L_q(\phi))$ topology on $L_p(\phi)$.

Ad (ii). Let $Y' \in L_q(\psi)$. Then, since the linear functional $\psi[\Theta(\cdot)BY']$ on $L_p(\phi)$ is continuous, there exists $X' \in L_q(\phi)$ such that $\psi[\Theta(Z)BY'] = \phi(ZX')$ for any $Z \in L_p(\phi)$. Since $AS = A$, we have $A_\lambda S = A_\lambda$. As $SX' \in L_1(\phi)$, $\phi(A_\lambda X') = \phi(A_\lambda SX')$ converges to $\phi(ASX') = \phi(AX')$, i.e., $\psi[\Theta(A_\lambda)BY']$ converges to $\psi[\Theta(A)BY']$. Hence $\Theta(A_\lambda)B \to \Theta(A)B$ with respect to the $\sigma(L_p(\psi),L_q(\psi))$ topology on $L_p(\psi)$. Similarly $B\Theta(A_\lambda) \to B\Theta(A)$ with respect to the same topolgoy on $L_p(\psi)$.

LEMMA 4. *Let $A,B \in \mathfrak{A} \cap L_p(\phi)$. Suppose $\Theta(A),\Theta(B) \in \mathfrak{B}$. Then $AB^* = B^*A = 0$ iff $\Theta(A)\Theta(B)^* = \Theta(B)^*\Theta(A) = 0$.*

PROOF. By Theorem 2(iii) of §2.

COROLLARY 1. *Let P, P_1 be projections in $\mathfrak{A} \cap L_1(\phi)$ with $\Theta(P),\Theta(P_1) \in \mathfrak{B}$.*

(i) *Suppose $P_1P = PP_1$. Then $|\Theta(P_1)|$ commutes with $|\Theta(P)|$ [$|\Theta(P)|$ denotes the absolute value of $\Theta(P)$]. Consequently, if $\Theta(P),\Theta(P_1) \geqslant 0$, then*

$$\Theta(P_1)\Theta(P) = \Theta(P)\Theta(P_1).$$

(ii) *Suppose $PP_1P = P_1$. Then $|\Theta(P_1)| \leqslant |\Theta(P)|$ and $|\Theta(P_1)| = Q|\Theta(P_1)|Q$ where Q denotes the support of $|\Theta(P)|$.*

PROOF (cf. [6, p. 95]). Ad (i). Let $Q_1 \equiv P_1P$, $Q_2 \equiv P - Q_1$, $Q_3 \equiv P_1 - Q_1$. Then Q_1, Q_2, Q_3 are mutually orthogonal projections with $Q_1 + Q_2 = P$ and $Q_1 + Q_3 = P_1$. A direct computation using Lemma 4 shows that

$$|\Theta(P)|^2|\Theta(P_1)|^2 = [\Theta(Q_1)^*\Theta(Q_1)]^2 = |\Theta(P_1)|^2|\Theta(P)|^2.$$

Ad (ii). Let $Q_0 \equiv P - P_1$. Then by Lemma 4 we have $|\Theta(P)|^2 = |\Theta(P_1)|^2 + |\Theta(Q_0)|^2$. The desired assertion then follows.

COROLLARY 2. *Suppose* $\Theta[\mathfrak{A}^+ \cap L_p(\phi)] \subset \mathfrak{B}^+$. *Let* $M_{r,\phi}$ *denote the two-sided ideal generated in* $\mathfrak{A}$ *by projections in* $L_1(\phi)$. *Let* $X,P \in M_{r,\phi}$ *with* P *a projection.*

(i) *Suppose* $XP = PX$. *Then* $\Theta(X)\Theta(P) = \Theta(P)\Theta(X)$.

(ii) *Suppose* $PXP = X$. *Then* $\Theta(X) = S\Theta(X)S$, *where* S *denotes the support of* $\Theta(P)$.

PROOF (cf. [**6**, p. 96]). Without loss of generality we can assume X to be positive. By making use of the spectral decomposition of X, Corollary 1 and Lemma 3 above, and the fact that the support of X belongs to $M_{r,\phi}$ and hence to $L_1(\phi)$ (cf. [**10**]), we establish (i). Statement (ii) is proved similarly.

LEMMA 5. *Suppose* $\Theta[\mathfrak{A}^+ \cap L_p(\phi)] \subset \mathfrak{B}^+$. *Then there exists a Jordan homomorphism* Φ' *of* $\overline{M}^u_{r,\phi}$ (*the closure of* $M_{r,\phi}$ *in* $\mathfrak{A}$ *with respect to the uniform topology*) *into* $\mathfrak{B}$ *such that*:

(i) $\Theta(X) = \Theta(P)\Phi'(X) = \Phi'(X)\Theta(P)$ *for each* $X \in M_{r,\phi}$ *and projection* $P \in M_{r,\phi}$ *with* $X = PXP$;

(ii) $\Phi'(P) = S_{\Theta(P)}$ [*the support of* $\Theta(P)$] *for each projection* $P \in M_{r,\phi}$.

PROOF. This generalizes Lemma 3 of [**6**]. The proof given there can be carried through with some modification. For completeness and clarity, however, we shall sketch the proof.

For X, P as given and for an arbitrary, strictly positive real number $\varepsilon > 0$, let $\Phi''_\varepsilon(X, P) \equiv [\varepsilon + \Theta(P)]^{-1}S\Theta(X)$, where S denotes the support of $\Theta(P)$. Then one can show that $0 \leqslant \Phi''_\varepsilon(X, P) \leqslant \|X\|$ in case $X \geqslant 0$, that in general $\Phi''(X, P) \equiv$ (strong) $\lim_{\varepsilon\to 0} \Phi''_\varepsilon(X, P)$ exists, and that the map $\Phi''(\cdot, P)$ from $P\mathfrak{A}P$ into $\mathfrak{B}$ is linear, bounded and positive. Moreover we have $\Phi''(X, P_1) = \Phi''(X, P)$ for any projection $P_1 \in \mathfrak{A}$ satisfying $S_X < P_1 < P$ ($S_X \equiv$ the support of X). Define, for $X \in M_{r,\phi}$, $\Phi''(X) \equiv \Phi''(X, S_X)$. One can show that the map Φ'' from $M_{r,\phi}$ into $\mathfrak{B}$ is linear, positive and bounded. Also $\Phi''(P) = S$ by considering $f_\varepsilon[\Theta(P)]$ where $f_\varepsilon(t) \equiv t(\varepsilon + t)^{-1}$. Define Φ' to be the unique continuous extension of Φ'' to $\overline{M}^u_{r,\phi}$. Φ' can be shown to be a Jordan homomorphism by means of the fact $\Phi'(Q_1)\Phi'(Q_2) = 0$ for any two mutually orthogonal projections $Q_1, Q_2 \in M_{r,\phi}$. (i) is proved by observing that, for positive X,

$$\begin{aligned}\Theta(P)\Phi''(X) &= \lim_{\varepsilon\to 0}\left[\Theta(P)\Phi''_\varepsilon(X, P)\right]\\ &= \lim_{\varepsilon\to 0}\left[\Phi''(P, P)\Theta(X)\right] = S\Theta(X) = \Theta(X),\end{aligned}$$

where $\lim_{\varepsilon\to 0}$ is taken in $L_p(\psi)$ with respect to the $\sigma(L_p(\psi), L_q(\psi))$ topology $(q^{-1} + p^{-1} = 1)$.

LEMMA 6. *Suppose* $\Theta[\mathfrak{A}^+ \cap L_p(\phi)] = \mathfrak{B}^+ \cap L_p(\psi)$. *Then*:

(i) *for any projection* $Q \in \mathfrak{B}$, *there exists a family of mutually orthogonal projections* $(P_\lambda)_{\lambda \in \Lambda}$ *in* $M_{r,\phi}$ *such that*

$$Q = \sum_{\lambda \in \Lambda} \Phi'(P_\lambda) \quad \text{and} \quad \Phi'(P_\lambda) \in M_{r,\psi};$$

(ii) *the weak closure of* $\Phi'(M_{r,\phi})$ *is* $\mathfrak{B}$.

PROOF. As (ii) follows readily from (i), and as each projection in $\mathfrak{B}$ is the direct sum of a family of mutually orthogonal projections in $M_{r,\psi}$, it suffices to prove (i) for a nonzero projection $Q \in M_{r,\psi}$. Then $\Theta^{-1}(Q) \in \mathfrak{A}^+ \cap L_p(\phi)$. By the spectral theorem there exist a positive number λ and a nonzero projection $P \in \mathfrak{A}$ such that $P \leq \lambda\Theta^{-1}(Q)$. Hence $P \in M_{r,\phi}$ and $\Theta(P) \leq \lambda Q$. Since $Q \in M_{r,\psi}$ we see that $\Theta(P)$ and its support $S_{\Theta(P)} = \Phi'(P)$ (by Lemma 5) belong to $M_{r,\psi}$; also $\Phi'(P) \leq Q$.

Now let $(P_n)_{n \in \mathbf{N}}$ be a maximal family of mutually orthogonal nonzero projections in $M_{r,\phi}$ such that $[\Phi'(P_n)]_{n \in \mathbf{N}}$ is a family of mutually orthogonal subprojections of Q with each $\Phi'(P_n) \in M_{r,\psi}$. The family is countable because ψ is faithful and $Q \in M_{r,\psi}$. Then $\Sigma_{n \in \mathbf{N}} \Phi'(P_n) = Q$. For if otherwise, $\bar{Q} \equiv Q - \Sigma_{n \in \mathbf{N}} \Phi'(P_n) \neq 0$. By the preceding paragraph, there is a nonzero projection $\bar{P} \in M_{r,\phi}$ such that $\Phi'(\bar{P}) \leq \bar{Q}$ and $\Phi'(\bar{P}) \in M_{r,\psi}$. Now as $S_{\Theta(\bar{P})}S_{\Theta(P_n)} = 0$, $\Theta(\bar{P})\Theta(P_n) = 0$ and $\bar{P}P_n = 0$. This contradicts the maximality of $(P_n)_{n \in \mathbf{N}}$; so $Q = \Sigma_{n \in \mathbf{N}} \Phi'(P_n)$.

LEMMA 7. *Suppose* $\Theta[\mathfrak{A}^+ \cap L_p(\phi)] = \mathfrak{B}^+ \cap L_p(\psi)$.

(i) *There exist two projections* Q_1, Q_2 *in the center of* $\mathfrak{B}$ *such that* $Q_1 + Q_2 = I$ *and such that the map* $X \mapsto \Phi'(X)Q_1$ [*resp*., $X \mapsto \Phi'(X)Q_2$] *of* $\overline{M}^u_{r,\phi}$ *into* $\mathfrak{B}$ *is a homomorphism* [*resp*., *an antihomomorphism*].

(ii) *For every pair of projections* Q_1, Q_2 *satisfying* (i), *for every* $X \in L_p(\phi)$ *and for every* $A \in \overline{M}^u_{r,\phi}$, *we have*

$$Q_1\Phi'(A)\Theta(X) = Q_1\Theta(AX) \quad [\text{resp.}, Q_2\Phi'(A)\Theta(X) = Q_2\Theta(XA)]$$

and

$$Q_1\Theta(X)\Phi'(A) = Q_1\Theta(XA) \quad [\text{resp.}, Q_2\Theta(X)\Phi'(A) = Q_2\Theta(AX)].$$

(iii) *For each pair of projections* Q_1, Q_2 *satisfying* (i), *there exists a pair of projections* P_1, P_2 *in the center of* $\mathfrak{A}$ *such that* $P_1 + P_2 = I$ *and such that, for every* $X \in L_p(\phi)$, *we have*

$$Q_1\Theta(X) = \Theta(P_1X) \quad [\text{resp.}, Q_2\Theta(X) = \Theta(P_2X)].$$

(iv) *The map* Φ' *from* $\overline{M}^u_{r,\phi}$ *into* $\mathfrak{B}$ *is ultraweakly continuous and extends uniquely to a Jordan isomorphism* Φ *of* $\mathfrak{A}$ *onto* $\mathfrak{B}$.

PROOF. This generalizes Lemma 5 of [**6**], and statements (i) and (ii) are proved as in [**6**] with obvious modification (e.g., for (ii), apply Proposition 5 of [**1**]).

Ad (iii). By Lemma 6, we may let $(P_\lambda)_{\lambda\in\Lambda}$ be a family of mutually orthogonal projections in $M_{r,\phi}$ such that $Q_1 = \Sigma_{\lambda\in\Lambda}\Phi'(P_\lambda)$. Put $P_1 \equiv \Sigma_{\lambda\in\Lambda} P_\lambda$. It is easy (cf. [6, p. 102]) to prove that for every $\lambda \in \Lambda$, $\Phi'(P_\lambda)\Theta(X) = \Theta(P_\lambda X)$, $X \in L_p(\phi)$. By Lemma 3, it follows that $Q_1\Theta(X) = \Theta(P_1X)$. Similarly we have $\Theta(X)Q_1 = \Theta(XP_1)$. As Q_1 is in the center of $\mathfrak{B}$, we have $\Theta(P_1Y) = \Theta(YP_1)$ for any $Y \in \mathfrak{A} \cap L_p(\phi)$. Thus P_1 belongs to the center of $\mathfrak{A}$. Similarly we prove that if $(P_\gamma)_{\gamma\in\Gamma}$ is a family of mutually orthogonal projections in $M_{r,\phi}$ such that $Q_2 = \Sigma_{\gamma\in\Gamma}\Phi'(P_\gamma)$, then $Q_2\Theta(X) = \Theta(XP_2)$, where $P_2 \equiv \Sigma_{\gamma\in\Gamma} P_\gamma$, and P_2 belongs to the center of $\mathfrak{A}$. Now

$$\Theta[X(P_1 + P_2)] = Q_1\Theta(X) + Q_2\Theta(X) = \Theta(X)$$

for any $X \in L_p(\phi)$, we conclude that $P_1 + P_2 = I$.

Ad (iv). In order to show that Φ' is ultraweakly continuous, it suffices (by Theorem 6 of [4]) to show that for every $Z \in L_1^+(\psi)$ the linear functional $A \mapsto \psi[\Phi'(A)Z]$ is ultraweakly continuous on $\overline{M}^u_{r,\phi}$. Since Θ is onto $L_p(\psi)$, it suffices, in turn, to show that for every $X \in L_p(\phi)$ and for every $W \in L_q(\psi)$ $(q^{-1} + p^{-1} = 1)$, the linear functional $A \mapsto \psi[\Phi'(A)\Theta(X)W]$ is ultraweakly continuous on $\overline{M}^u_{r,\phi}$. By (ii) and (iii) above, it is easy to see that $\Phi'(A)\Theta(X) = \Theta(P_1AX + P_2XA)$. Hence

$$\psi[\Phi'(A)\Theta(X)W] = \psi[\Theta(P_1AX + P_2XA)W].$$

Since the linear functional $\psi[\Theta(\cdot)W]$ is continuous on $L_p(\phi)$, there exists $V \in L_q(\phi)$ such that $\psi[\Theta(Y)W] = \phi(YV)$ for every $Y \in L_p(\phi)$. Hence we have

$$\begin{aligned}\psi[\Phi'(A)\Theta(X)W] &= \phi[(P_1AX + P_2XA)V]\\ &= \phi(AXVP_1) + \phi(AVP_2X).\end{aligned}$$

Since XVP_1 and $VP_2X \in L_1(\phi)$, the maps $A \mapsto \phi(AXVP_1)$ and $A \mapsto \phi(AVP_2X)$ are ultraweakly continuous. Thus Φ' is ultraweakly continuous on $\overline{M}^u_{r,\phi}$.

The rest of the proof is standard (cf. [6, p. 103]).

THEOREM 3. *Suppose* $\Theta[\mathfrak{A}^+ \cap L_p(\phi)] = \mathfrak{B}^+ \cap L_p(\psi)$, $p \in [1, \infty)$. *Then there exist uniquely a Jordan *-isomorphism* Φ *of* $\mathfrak{A}$ *onto* $\mathfrak{B}$, *and a positive selfadjoint operator* $Z \in L_p(\psi)$ *affiliated with the center of* $\mathfrak{B}$ *such that*

$$\Theta(X) = \Phi(X)Z, \qquad X \in \mathfrak{A} \cap L_p(\phi).$$

Suppose, in addition, that $\mathfrak{A}$ *or* $\mathfrak{B}$ *is a factor. Then* Φ *is an isomorphism or an anti-isomorphism of* $\mathfrak{A}$ *onto* $\mathfrak{B}$, *and* Z *is proportional to the identity of* $\mathfrak{B}$; *so there exists uniquely a strictly positive number* λ *such that* $\Theta(X) = \lambda\Phi(X)$ *for every* $X \in \mathfrak{A} \cap L_p(\phi)$.

PROOF. This "generalizes" Theorem 1 of [6]. The proof there carries through, with some modification, in the present context. For completeness and clarity, however, we shall sketch the proof.

Let $(P_\lambda)_{\lambda\in\Lambda}$ be a family of mutually orthogonal projections in $M_{r,\phi}$ such that $I = \sum_{\lambda\in\Lambda} P_\lambda$. Let S_λ be the support of $\Theta(P_\lambda)$. Let Z be the closure of the "direct sum" Z' of $\Theta(P_\lambda)|_{S_\lambda(\kappa)}$ (κ denotes the Hilbert space on which $\mathfrak{B}$ acts). Then it follows from Lemmas 3 and 7 that

$$\Theta(X) = \sum_{\lambda\in\Lambda} \Theta(XP_\lambda) = \sum_{\lambda\in\Lambda} \Phi(X)\Theta(P_\lambda),$$

where $\sum_{\lambda\in\Lambda}$ is taken in $L_p(\psi)$ with respect to the $\sigma(L_p(\psi), L_q(\psi))$ topology $(q^{-1} + p^{-1} = 1)$. Since

$$\Phi(X)\Theta(P_{\lambda_0}) = \left[\sum_{\lambda\in\Lambda} \Phi(X)\Theta(P_\lambda)\right]S_{\lambda_0} = \Theta(X)S_{\lambda_0},$$

it follows that $\Phi(X)Z' \subset \Theta(X)$ and, taking adjoints, it can be deduced that $\Theta(X) = Z\Phi(X) = \Phi(X)Z$. As the composite $\Phi(X) \circ Z$ is contained in the composite $Z \circ \Phi(X)$, Z is seen to be affiliated with the center of $\mathfrak{B}$. Φ is unique because $\Phi(P)$ has to be the support of $\Theta(P)$ for each projection $P \in M_{r,\phi}$. To see the uniqueness of Z, let Z_1 be another operator having the same properties as Z. If $Z_1 - Z \neq 0$, then by the spectral theorem and Lemma 6, there would be a projection $P \in M_{r,\phi}$ such that $(Z_1 - Z)\Phi(P) \neq 0$. Since $Z_1\Phi(P) = \Theta(P) = Z\Phi(P)$, we have $Z_1 = Z$. The rest of the theorem follows from well-known results on Jordan isomorphism (cf. [11]). (In the present context, Lemma 4 with $p = 2$ is Lemma 2 of [6], so the above outline works for $p = 2$ too.)

COROLLARY 1. *Let $\mathfrak{A}$ be a finite factor with the canonical trace ϕ, and let Θ be a linear isometry from $L_p(\phi)$ $(1 \leqslant p < \infty)$ onto itself with $\Theta(\mathfrak{A}^+) = \mathfrak{A}^+$. Then the restriction of Θ to $\mathfrak{A}$ is either an automorphism or an anti-automorphism of $\mathfrak{A}$.*

COROLLARY 2. *Let $\mathfrak{A}$ be a type* I *factor with the canonical trace ϕ, and let Θ be a linear isometry from $L_p(\phi)$ $(1 \leqslant p < \infty)$ onto itself with $\Theta[L_p^+(\phi)] = L_p^+(\phi)$. Then there exists an automorphism or an anti-automorphism Φ of $\mathfrak{A}$ such that $\Theta(X) = \Phi(X)$ for every $X \in L_p(\phi)$.*

PROOF. These corollaries follow readily from Theorem 3.

COROLLARY 3 (cf. [5, Theorem 2]). *Let Γ be a linear map from $\mathfrak{A}$ onto $\mathfrak{B}$. Suppose for some $p \in [1, \infty)$, $\Gamma[\mathfrak{A} \cap L_p(\phi)] = \mathfrak{B} \cap L_p(\psi)$ and suppose that $\|\Gamma(A)\|_p = \|A\|_p$ for every $A \in \mathfrak{A} \cap L_p(\phi)$. Then the following statements are equivalent*:

(i) Γ *is a Jordan $*$-isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$*;

(ii) $\Gamma(\mathfrak{A}^+) = \mathfrak{B}^+$ *and* $\Gamma(I) = I$.

PROOF. Γ can be extended to a linear isometry from $L_p(\phi)$ onto $L_p(\psi)$. Thus (ii) $\Rightarrow$ (i) by Theorem 3 (recall that Φ is ultraweakly continuous). The implication (i) $\Rightarrow$ (ii) is well known [**11**].

LEMMA 8. *Suppose there is a family $(P_\lambda)_{\lambda\in\Lambda}$ of mutually orthogonal projections in $\mathfrak{A}$ such that $\Sigma_{\lambda\in\Lambda} P_\lambda = I$ and $\phi(P_\lambda) < \infty$, $\Theta(P_\lambda) \in \mathfrak{B}$ for each $\lambda \in \Lambda$. Then there exists a partial isometry $V \in \mathfrak{B}$ such that for any projection $P \in \mathfrak{A}$ with $\Theta(P) \in \mathfrak{B}$ and $P \leqslant P_{\lambda_0}$ (for some $\lambda_0 \in \Lambda$), we have $\Theta(P)V \geqslant 0$.*

PROOF. Denote $\Theta(P_\lambda) \equiv U_\lambda$, let $U_\lambda = W_\lambda|U_\lambda|$ be its polar decomposition, and let F_λ [resp., E_λ] be the initial [resp., final] projection of the partial isometry W_λ. By Lemma 4 above, $U_\lambda U_{\lambda'}^* = U_{\lambda'}^* U_\lambda = 0$ $(\lambda \neq \lambda')$; so U_λ, $U_{\lambda'}$ (resp., U_λ^*, $U_{\lambda'}^*$) have mutually orthogonal supports. Thus $E_\lambda E_{\lambda'} = 0$, $F_\lambda F_{\lambda'} = 0$. Let $V \equiv \Sigma_{\lambda\in\Lambda} W_\lambda^*$. Clearly $V \in \mathfrak{B}$ is a partial isometry. Let $\lambda_0 \in \Lambda$. Then $\Theta(P_{\lambda_0})V = U_{\lambda_0}V = |U_{\lambda_0}^*| \geqslant 0$ as $U_{\lambda_0}(1 - F_{\lambda_0}) = 0$. Let P be a projection in $\mathfrak{A}$ majorised by P_{λ_0}, and let $Q \equiv P_{\lambda_0} - P$. Then $\Theta(P) + \Theta(Q) = U_{\lambda_0}$ and $\Theta(Q)^*\Theta(P) = 0$. Thus $[\Theta(Q)V]^*$ and $[\Theta(P)V]^*$ have mutually orthogonal supports. Since $|U_{\lambda_0}^*| = [\Theta(P)V]^* + [\Theta(Q)V]^*$ is nonnegative, we conclude that $\Theta(P)V \geqslant 0$.

PROPOSITION 1. *Suppose $\mathfrak{A}$ is finite (ϕ may be semifinite) and $\Theta[\mathfrak{A} \cap L_p(\phi)] = \mathfrak{B} \cap L_p(\psi)$. Then there is a partial isometry $V \in \mathfrak{B}$ such that $\Gamma(\cdot) \equiv \Theta(\cdot)V$ is a linear isometry from $L_p(\phi)$ into $L_p(\psi)$ with $\Gamma[\mathfrak{A}^+ \cap L_p(\phi)] \subset \mathfrak{B}^+ \cap L_p(\psi)$, and $VV^* = I$.*

PROOF. By Proposition 10 of [**9**, §6, p. 100], there exists a family $(P_\lambda)_{\lambda\in\Lambda}$ of central projections in $\mathfrak{A}$ satisfying the conditions of Lemma 8. Let V be constructed as in Lemma 8.

For $X \in M_{r,\phi}$ and $\lambda \in \Lambda$, let $X_\lambda \equiv P_\lambda X P_\lambda$, and let F_λ be the initial projection of W_λ introduced in the proof of Lemma 8. Then $X =$ (weak) $\Sigma_{\lambda\in\Lambda} X_\lambda$ and $VV^*\Theta(X_\lambda)^* = VV^*F_\lambda\Theta(X)^*$ (by Corollary 2 of Lemma 4) $= \Theta(X_\lambda)^*$. By Lemma 3, $VV^*\Theta(X)^* = \Theta(X)^*$. Since

$$\Theta\big[\mathfrak{A} \cap L_p(\phi)\big] = \mathfrak{B} \cap L_p(\psi)$$

and since $M_{r,\phi}$ is dense in $L_p(\phi)$, we conclude that $VV^* = I$. As

$$\|\Theta(X)V\|_p = \|V^*\Theta(X)^*\|_p = \|\Theta(X)^*\|_p = \|X\|_p,$$

$\Gamma(\cdot) \equiv \Theta(\cdot)V$ is a linear isometry from $L_p(\phi)$ into $L_p(\psi)$. By Lemma 8, it is clear that $\Gamma[\mathfrak{A}^+ \cap L_p(\phi)] \subset \mathfrak{B}^+ \cap L_p(\phi)$.

The following theorem "generalizes" Theorem 1 of [**5**] with a completely different proof.

THEOREM 4. *Suppose* $\mathfrak{A}$, $\mathfrak{B}$ *are finite* (ϕ, ψ *may be semifinite*) *and* $\Theta[\mathfrak{A} \cap L_p(\phi)] = \mathfrak{B} \cap L_p(\psi)$, $p \in [1, \infty)\setminus\{2\}$. *Then there are a Jordan *-isomorphism* Φ *of* $\mathfrak{A}$ *onto* $\mathfrak{B}$, *a positive selfadjoint operator* $Z \in L_p(\psi)$ *affiliated with the center of* $\mathfrak{B}$, *and a unitary* $W \in \mathfrak{B}$ *such that* $\Theta(X) = \Phi(X)ZW$, $X \in \mathfrak{A} \cap L_p(\phi)$.

PROOF. Since $\mathfrak{B}$ is finite, the partial isometry V in the preceding proposition is unitary (cf. [**9**, p. 217]) and $\Gamma(\cdot) \equiv \Theta(\cdot)V$ is onto $L_p(\psi)$ with

$$\Gamma[\mathfrak{A}^+ \cap L_p(\phi)] = \mathfrak{B}^+ \cap L_p(\psi).$$

A direct application of Theorem 3 then completes the proof.

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